

Finite Element Modelling with Transformation Techniques

F. Henrotte, B. Meys, H. Hedia, P. Dular* and W. Legros

Université de Liège - Dept. of Electrical Engineering - Institut Montefiore - Sart Tilman Bât B28 - B-4000 Liège (Belgium)

Abstract—Transformation methods are a very powerful tool in finite element modelling. In many cases, an adequate mapping transforms the problem into an easier one or allows to take advantage of the symmetries. This paper demonstrates that any mapping can be handled automatically provided the classical vector analysis approach is given up for the benefit of a differential geometry approach. As a first example, it is shown that axisymmetrical problems need no more a particular treatment provided the mapping of the cylindrical coordinates on the cartesian ones is considered as it is. Furthermore, a novel axisymmetrical formulation is proposed which relies on one further transformation and improves considerably the quality of the interpolated field. Transformation methods are also of great help to model the infinite space by means of finite elements. Many authors have presented such transformations which are often instances of the same general shell transformation that is presented here.

Index terms—Transforms, differential geometry, TEAM Workshop problem 11, object oriented programming, software design/development, Interpolation.

I. INTRODUCTION

Transformation methods are a very powerful tool in finite element modelling. An adequate mapping transforms the problem into an easier one or allows to take advantage of the symmetries. Transformation methods are of great help to model the infinite space by means of finite elements and many authors have presented such transformations [1][2]. It can also be shown that axisymmetrical problems need not a particular treatment provided the mapping of the cylindrical coordinates on the cartesian ones is considered as it is.

From a general point of view, the finite element method is a mathematical tool to discretize a continuous problem and obtain an algebraic system $Ax = b$. The components of the matrix A are the result of the integration of some densities on the finite elements. To make this integration easier, it is customary to define each finite element V of the mesh as the image of a reference element V_{ref} by a mapping ϕ_{ref} involving the coordinates of its N nodes. The same reference element is shared by all the elements of the mesh that have the same shape. It is the domain on which are defined the basis functions s_α for the interpolation of the discretized field as well as the gauss points and the weights that are needed for a numerical integration. It is therefore the domain over which the integration is performed.

Actually, everything is ready in most of the finite element programs to work with transformation methods because this

mapping ϕ_{ref} at least is always present. Working with transformation methods means nothing more than combining it with one or several other changes of coordinates.

For an automatic consideration of any transformation, the expressions of the finite element densities must be formally identical whatever the coordinate system. Classical vector analysis fails to give such expressions because it assumes orthonormal coordinates only and ignores then the geometrical notions of *metric* and *volume form* which happen to be trivial in that particular case. In order to get coordinate independent expressions, it is necessary to work with true coordinate systems (which are not always orthonormal) and to consider the geometrical notions of *metric* and *volume form*. Those differential geometry notions are intrinsic, i.e. peculiar to each coordinate system (while the Jacobian matrix is rather a kind of bridge between two coordinates systems), and the densities expressed in terms of them are therefore automatically coordinate independent.

II. MAPPINGS

Let M and N be two spaces and $\phi : M \mapsto N$ a nonsingular mapping. Let $F(M)$ and $F(N)$ be the sets of the fields defined on M and N respectively.

The general rules for mapping the fields is just what is needed to get coordinate independent expressions for the finite element densities. For that purpose, the fields have to be sorted out into four categories in accordance with the derivative operator they are subjected to in the physical laws.

$$(F^0) \xrightarrow{grad} (F^1) \xrightarrow{curl} (F^2) \xrightarrow{div} (F^3) \quad (1)$$

The elements of the $\{F^k, k = 0, 1, 2, 3\}$ sets are respectively noted $\omega^{(k)}$. The fields in F^0 are subjected to the *grad* operator (e.g. temperature, electric potential). Those in F^3 are densities (e.g. charge density). Both are scalar fields. The fields in F^1 are subjected to the *curl* operator (e.g. magnetic field, electric field, vector potential) and are noted with one lower indice, $(\omega^{(1)})_i$. The fields in F^2 are subjected to the *div* operator (e.g. induction field, current density) and are noted with one upper indice, $(\omega^{(2)})^i$. These two are vector fields. Note that the position of the indice is important since it determines how the fields transform.

The integration of a density $f \in F^3(N)$ on $N = \phi(M)$ can be performed on M thanks to the rule

$$\int_{\phi(M)} f = \int_M \phi^*(f) \quad (2)$$

where the map $\phi^* : F(N) \mapsto F(M)$ is called the dual map of ϕ . It is known if ϕ is known as one will see further on.

If the coordinates $\{X^I, I = 1, 2, 3\}$ are defined on N and the coordinates $\{u^j, j = 1, 2, 3\}$ on M , the mapping ϕ and its

Manuscript received June 3, 1998

F. Henrotte, fax +32-4-3662910, F. Henrotte@ulg.ac.be,

<http://www.montefiore.ulg.ac.be/services/elap/Welcome.html>

*Research Associate with the Belgian National Fund for Scientific Research.

This text presents research results of the Belgian programme on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming.

jacobian matrix Λ_j^I are defined by

$$X^I = X^I(u^j) \quad \Lambda_j^I(u^k) = \frac{\partial X^I}{\partial u^j}(u^k). \quad (3)$$

The inverse of Λ_j^I is noted Λ_I^j .

III. METRIC MATRIX

The metric matrix is a 3x3 symmetric matrix. It is noted with two lower indices, g_{ij} . In an euclidean space E the metric is the unity matrix $g_{IJ} = \delta_{IJ}$. In any system of coordinates, the components of the metric matrix are computed thanks to the jacobian matrix of the mapping from the euclidean space E onto that system of coordinates by the relation $g_{ij} = \Lambda_i^I g_{IJ} \Lambda_j^J$ where the implicit summation on repeated indices, one upper and one lower, is assumed as it will be throughout the paper. Its determinant is the square of the determinant of the Jacobian matrix: $\det(g_{ij}) \equiv g = \Lambda^2 \equiv (\det(\Lambda_j^I))^2$. The inverse of the metric matrix is noted with two upper indices and is given by $g^{ij} = \Lambda_i^I g^{IJ} \Lambda_j^J$.

IV. MULTIPLE MAPPINGS

The mapping ϕ can be the combination of several successive mappings (Fig. 1). Thanks to the chain rule of partial derivatives, it is clear that, if the jacobian matrices of the individual mappings are given, the resultant jacobian matrix is simply their matrix product:

$$\Lambda_k^I \equiv \frac{\partial X^I}{\partial u^k} = \frac{\partial X^I}{\partial x^i} \frac{\partial x^i}{\partial y^j} \frac{\partial y^j}{\partial u^k} \equiv \Lambda_i^I(x^i) \Lambda_j^i(y^j) \Lambda_k^j(u^k). \quad (4)$$

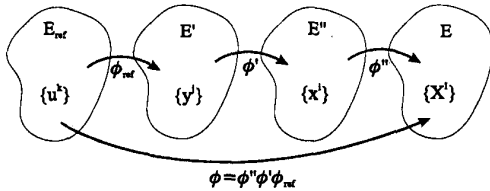


Fig. 1. Multiple mappings

V. TRANSFORMATION METHODS

In the finite element method, the ends of the chain of spaces (Fig. 1) are always the same. The first space is the reference space E_{ref} where the integrations are performed and the shape function defined. The last one is the physical space E in which the physical laws are given. The reference space is either 1D, 2D or 3D in accordance with the symmetries of the system. The physical space is always an euclidean one.

A third space of importance is the mesh space E' . It is the space spanned by the finite elements defined in the data base of the problem. If no transformation method is considered, the mesh space is identical with the physical space, $E' \equiv E$. The mapping $\phi_{ref} : E_{ref} \mapsto E'$ and its jacobian matrix can be written:

$$y^j(u^k) = \sum_{\alpha=1}^N N_{\alpha}^j s_{\alpha}(u^k) \quad \Lambda_k^j = \sum_{\alpha=1}^N N_{\alpha}^j \frac{\partial s_{\alpha}}{\partial u^k} \quad (5)$$

where N_{α}^j are the coordinates of the N nodes of the finite element and s_{α} the corresponding nodal shapes functions.

Fundamentally, the principle of a transformation technique is to map the shape functions, which are simple polynomials in E_{ref} , onto interpolation functions in E that are not necessarily polynomials in order to fit better the interpolated field in that space. One or several intermediary mappings (two in Fig. 1) can be defined for that purpose giving rise to one or several intermediary spaces: E', E'', \dots

The following tables give the rules to map the fields and the rules to compute finite element densities that are the product of two fields, say η and ξ , and a physical characteristic function μ . This is enough for most of the F.E. densities encountered in a wide variety of formulations.

TABLE I
MAPPING OF THE DIFFERENT KIND OF FIELDS

	$F(N) \mapsto F(M)$	$F(M) \mapsto F(N)$
F^0	$(\phi^* \omega) = \omega$	$(\phi^{-*} \omega) = \omega$
F^1	$(\phi^* \omega)_i = \Lambda_i^I \omega_I$	$(\phi^{-*} \omega)_I = \Lambda_I^i \omega_i$
F^2	$(\phi^* \omega)^i = \sqrt{g} \Lambda_i^I \omega^I$	$(\phi^{-*} \omega)^I = \Lambda_I^i \omega^i / \sqrt{g}$
F^3	$(\phi^* \omega) = \sqrt{g} \omega$	$(\phi^{-*} \omega) = \omega / \sqrt{g}$

TABLE II
COORDINATE INDEPENDENT EXPRESSIONS FOR THE MOST COMMON FINITE ELEMENT DENSITIES

$\eta \backslash \xi$	F^0	F^1	F^2	F^3
F^0	$\mu \eta \xi \sqrt{g}$			$\mu \eta \xi$
F^1		$\mu \eta_i g^{ij} \xi_j \sqrt{g}$	$\mu \eta_i \xi^i$	
F^2		$\mu \eta^i \xi_i$	$\mu \eta^i g_{ij} \xi^j / \sqrt{g}$	
F^3	$\mu \eta \xi$			$\mu \eta \xi / \sqrt{g}$

Those expressions are valid in any coordinate system. Everything concerning the mappings is gathered in the computation of the metric matrix and its determinant. With a careful programming, made easier within an object oriented approach [3], the finite elements involved in different transformations only differ by the subroutine that computes the components of the metric matrix. All the remainder is strictly unchanged. In other words, only the mapping itself must be defined to work a new transformation out. It is of course the bare minimum.

VI. APPLICATIONS

A. Axisymmetrical problems

Axisymmetrical problems are just problems mapped from the cartesian coordinates of E onto cylindrical coordinates. The mappings ϕ_{ref} and ϕ' and the jacobian matrix of ϕ' are expressed by (7) with $\{X^I = X, Y, Z\}$, $\{y^j = r, z, \theta\}$ and $\{u^k = u, v, w\}$.

$$\begin{cases} r = \sum N_{\alpha}^r s_{\alpha}(u, v) \\ z = \sum N_{\alpha}^z s_{\alpha}(u, v) \\ \theta = w \end{cases} \quad \begin{cases} X = r \cos \theta \\ Y = z \\ Z = r \sin \theta \end{cases} \quad (6)$$

$$\Lambda_j^I(y^j) = \begin{pmatrix} \cos \theta & 0 & -r \sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & r \cos \theta \end{pmatrix} \quad (7)$$

The components of the vector potential $a \in F^1$ are (a_X, a_Y, a_Z) in E and (a_r, a_z, a_θ) in E' . Since we are now working with true coordinates and have given up the classical Vector Analysis, the *curl* operator has the same expression in any coordinate system. The components of the induction field in E and E' are thus respectively given by (8).

$$\begin{aligned} (b^X, b^Y, b^Z) &= \left(\frac{\partial a_Z}{\partial Y} - \frac{\partial a_Y}{\partial Z}, \frac{\partial a_X}{\partial Z} - \frac{\partial a_Z}{\partial X}, \frac{\partial a_Y}{\partial X} - \frac{\partial a_X}{\partial Y} \right) \\ (b^r, b^z, b^\theta) &= \left(\frac{\partial a_\theta}{\partial z} - \frac{\partial a_z}{\partial \theta}, \frac{\partial a_r}{\partial \theta} - \frac{\partial a_\theta}{\partial r}, \frac{\partial a_z}{\partial r} - \frac{\partial a_r}{\partial z} \right) \end{aligned} \quad (8)$$

The axial symmetry implies that nothing depends on θ , the model can be restricted to the plane $\theta = 0$. In that case, the vector potential reduces to its θ -component $a_\theta(r, z)$, the jacobian matrix of ϕ' is $\text{diag}(1, 1, r)$ and its determinant equals r . If a linear interpolation $a_\theta = cr + d$ is used, the z -component of the induction is $b^z = -c$ by (8) and its Y -component is $b^Y = -c/\sqrt{g}$ as seen in Table II. Since $g = r^2$ in the coordinates $\{X^I\}$, it involves a r^{-1} term that causes a terrible jagged pattern for the induction field (Fig. 2).

To avoid it, the solution is to make one more change of coordinate in order to use $\rho = r^2$ instead of r as the radial coordinate. The mappings ϕ_{ref} , ϕ' and ϕ'' , expressed with $\{X^I = X, Y, Z\}$, $\{y^j = \rho, z, \theta\}$ and $\{x^i = r, z, \theta\}$ are then given by (9) with $N_\alpha^\rho = (N_\alpha^r)^2$. The jacobian matrix of $\phi'\phi''$ is $\text{diag}(1/2\sqrt{\rho}, 1, \sqrt{\rho})$ and $g = 1/4$ is a constant in E' . A linear interpolation $a_\theta = c\rho + d$ gives now a constant induction field $b^Y = -2c$. This is exactly the gentle property one has in the planar case.

$$\begin{cases} \rho = \sum N_\alpha^\rho s_\alpha(u, v) \\ z = \sum N_\alpha^z s_\alpha(u, v) \\ \theta = w \end{cases} \quad \begin{cases} r = \sqrt{\rho} \\ z = z \\ \theta = \theta \end{cases} \quad \begin{cases} X = r \cos \theta \\ Y = z \\ Z = x \sin \theta \end{cases} \quad (9)$$

Figs. 2 and 3 show the induction field computed for the Problem 11 of TEAM Workshop. The problem consists of a hollow, nonmagnetic, conducting sphere ($\rho = 2.10^{-9} \Omega m$, $r_{int} = 50mm$ and $r_{ext} = 55mm$) in a spatially uniform induction field that is instantaneously switched on at $t = 0$. The Y -component of the induction field computed with first order finite elements is presented on two cuts, respectively along X -axis and Y -axis, and for different times ($t=1, 2, 5, 10$ and $20ms$). The classical axisymmetrical formulation exhibits a very poor representation of the induction field which becomes moreover dramatically unaccurate near the axis.

B. Shell transformation

In electromagnetism, the boundary conditions must be imposed at the infinity and many transformation methods have been proposed to model the infinite space by means of finite elements. Very often, the transformations proposed are instances of the general shell transformation given by

$$(X^I - C^I) = (y^j - C^j) \delta_j^I F(A, B, r(y^j)), \quad (10)$$

$$F(A, B, r) = \left(\frac{A(B - A)}{r(B - r)} \right)^p, \quad (11)$$

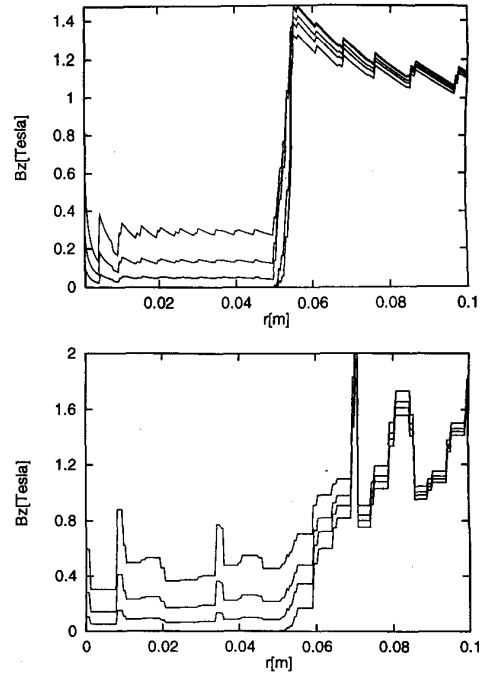


Fig. 2. $B_Y(t=1, 2, 5, 10, 20ms)$ along X-axis (above) and Y-axis (below) with the classical axisymmetric formulation.

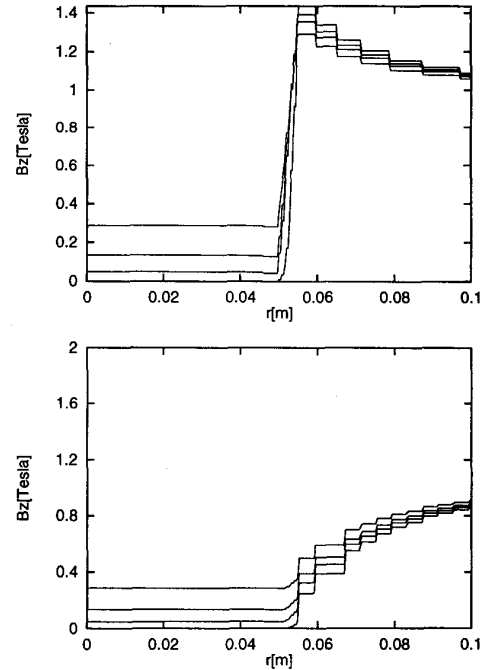


Fig. 3. $B_Y(t=1, 2, 5, 10, 20ms)$ along X-axis (above) and Y-axis (below) with the modified axisymmetric formulation.

$$\frac{dF}{dr} = -\theta \frac{F}{r} \quad \theta = \frac{B-2r}{p(B-r)} \quad (12)$$

with $\{X^I = X, Y, Z\}$ and $\{y^j = x, y, z\}$, C^I the fixed point of the transformation ($C^I = C^j \delta_j^I$), A and B the inner and outer "radius" of the shell.

This transformation applies to cylindrical shells, paralelipedic shells along the k^{th} direction and to spherical shells with respectively

$$\begin{aligned} r(y^j) &= \sqrt{(x - C^x)^2 + (y - C^y)^2}, \\ r(y^j) &= (y^k - C^k), \\ r(y^j) &= \sqrt{(x - C^x)^2 + (y - C^y)^2 + (z - C^z)^2}. \end{aligned}$$

The jacobian matrix of this mapping is given by (13) with $n^i = \frac{y^i - C^i}{r}$ and its determinant is $\Lambda_{shell}(A, B, r) = F^2(1 - \theta)$.

$$\Lambda_j^I = F \begin{pmatrix} 1 - \theta n^x \frac{\partial r}{\partial x} & -\theta n^x \frac{\partial r}{\partial y} & -\theta n^x \frac{\partial r}{\partial z} \\ -\theta n^y \frac{\partial r}{\partial x} & 1 - \theta n^y \frac{\partial r}{\partial y} & -\theta n^y \frac{\partial r}{\partial z} \\ -\theta n^z \frac{\partial r}{\partial x} & -\theta n^z \frac{\partial r}{\partial y} & 1 - \theta n^z \frac{\partial r}{\partial z} \end{pmatrix} \quad (13)$$

In order to have a regular mapping, the determinant of the jacobian matrix of the shell transformation must be different from zero everywhere. It is also natural for this particular transformation to impose that no element of the shell is contracted and that they are all expanded. That means that $\Lambda_{shell} \geq 1$, so $\theta \geq 0$. Finally, the continuity of the metric is ensured by $\Lambda_{shell}(A, B, r = A) = 1$. That means that the distortion of the finite elements varies smoothly with no discontinuity across the inner boundary of the shell. Those conditions are all verified if the outer radius of the shell is chosen twice the inner radius: $B = 2A$.

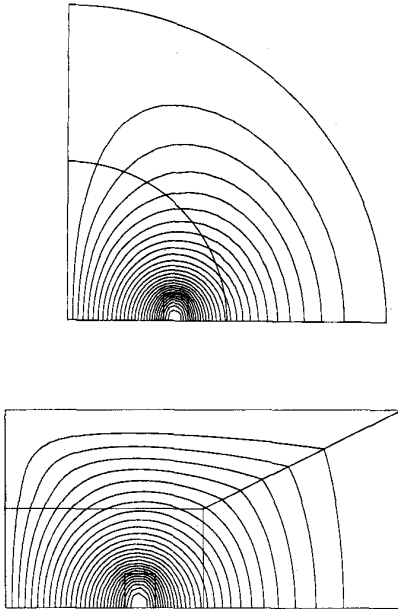


Fig. 4. Field lines with a cylindrical and a rectangular shell.

Fig. 4 shows the field lines obtained with a cylindrical and a rectangular shell for an axisymmetric coil. Exactly as it is

for nontransformed elements, the accuracy of the transformed elements is related to the quality of the discretisation. The inductances computed with this technique have shown a good agreement with measurements.

The metric matrix components tends towards infinity on the outer boundary of the shell. This requires some numerical precautions to avoid floating point operation errors, especially for the visualization of the computed fields. An other solution is to impose that the image of the outer boundary of the shell is at a finite distance: $F(A, B, r = B) \Rightarrow L < \infty$. The relations (11) and (12) remain true with the simple substitution

$$\begin{cases} A' = A \\ B' = \frac{LB^2 - A^2}{LB - A} \end{cases} \quad (14)$$

and one can check that

$$\lim_{L \rightarrow \infty} B' = B. \quad (15)$$

The value of the parameter p has to be chosen in accordance with the characteristics of the problem under consideration. For instance, $R(X^I) = r(y^j)F(r(y^j))$ by (10) where $R(X^I)$ is the function $r(y^j)$ expressed in terms of the coordinates $\{X^I\}$. The shell transformation (10) (11) has then the properties to map R^{-q} fields onto $r^{q(p-1)}(B-r)^{qp}$ fields that are polynomials if p is integer and $p \geq 1$. In the axisymmetric case, the far vector potential field behaves like R^{-2} . Second order polynomial in r can thus match the field in the shell if we choose $p = 1$.

VII. CONCLUSION

A general presentation of transformation methods has been given in this paper. The technique is a very flexible tool if the finite element densities are written in a coordinate independent fashion. Such ready-to-use expressions have been given which translate into slight modifications of existing codes. Only the mapping itself (which can be a combination of any number of individual mappings) has to be implemented and all the remainder of the program is unchanged. Treated in this way, transformation methods applies automatically to any formulation and any shape functions. The benefit of multiple mappings has been shown in a few examples making use of up to four successive mappings in the case of the modified axisymmetrical shell.

REFERENCES

- [1] D.A. Lowther, E.M. Freeman, B. Forghani, "A sparse matrix open boundary method for finite element analysis," *IEEE Transactions on Magnetics*, vol. 25, No. 4, pp. p 2810-2812, 1989.
- [2] J.F. Imhoff, G. Meunier, X. Brunotte, J.C. Sabonnadière, "An original solution for unbounded electromagnetic 2D- and 3D- problems throughout the finite element method" *IEEE Transactions on Magnetics*, vol. 26, No. 5, pp. p 1629-1661, 1990.
- [3] F. Henrotte, B. Meys, A. Genon, W. Legros, "An Object Oriented Decomposition of the F.E. procedure," *IEEE Transactions on Magnetics*, vol. 32, No. 3, pp. 1441-1444, 1996.
- [4] F. Henrotte, H. Hedia, et al., "A new method for axisymmetrical linear and non linear problems," *IEEE Transactions on Magnetics*, vol. 29, No. 2, pp. 1352-1355, 1993.
- [5] J.B.M. Melissen, J. Simkin, "A new coordinate transform for the finite element solution of axisymmetrical problems in Magnetostatics" *IEEE Transactions on Magnetics*, vol. 26, No. 2, pp. p 391-394, 1990.