3-D electromagnetic wave scattering

Jouni Mäkitalo

October 23, 2014

Consider a homogeneous dielectric (lossy) non-magnetic compact scatterer with domain $\Omega \subset \mathbb{R}^3$ that has permittivity $\epsilon \in \mathbb{C}$ that is ϵ_0 in the complement region Ω_c. The complement region has boundary $(-\partial\Omega) \cup \Gamma$, where Γ is a spherical surface enclosing Ω. The scattering problem is formulated via the electric Hemholtz equation:

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = \mathbf{0} \qquad \text{in } \Omega \cup \Omega_c \tag{1}$$

$$(\nabla \times \mathbf{E}) \times \mathbf{n} - ik\mathbf{E} = \mathbf{g} \qquad \text{on } \Gamma, \tag{2}$$

where $\mathbf{g} = i\omega\mu\mathbf{H}_0 \times \mathbf{n} - ik\mathbf{E}_0$ with incident fields $\mathbf{E}_0, \mathbf{H}_0$. The boundary condition on Γ approximates the Silver-Müller radiation condition:

$$\lim_{|\mathbf{r}|\to\infty} (\mathbf{E} - \mathbf{E}_0) \times \mathbf{r} + |\mathbf{r}|\eta(\mathbf{H} - \mathbf{H}_0) = \mathbf{0}. \tag{3}$$

The solution space $\mathbf{X}$ consists of square-integrable functions $\mathbf{x}$ with square-integrable curls. The functions are tangential on Γ. We test Eq. (1) by $\mathbf{w} \in \mathbf{X}$ to obtain

$$\int_{\Omega\cup\Omega_c} \mathbf{w} \cdot \nabla \times \nabla \times \mathbf{E} dV - \int_{\Omega\cup\Omega_c} k^2\mathbf{w} \cdot \mathbf{E} dV = 0 \tag{4}$$

for all $\mathbf{w}$. By the use of the Green formula

$$\int_V \mathbf{w} \cdot \nabla \times \mathbf{v} - \mathbf{v} \cdot \nabla \times \mathbf{w} dV = \int_{\partial V} \mathbf{w} \times \mathbf{n} \cdot \mathbf{v}_t dS \tag{5}$$

we obtain

$$\int_{\Omega\cup\Omega_c} \nabla \times \mathbf{w} \cdot \nabla \times \mathbf{E} dV + \int_\Gamma \mathbf{w} \times \mathbf{n} \cdot \nabla \times \mathbf{E} dS - \int_{\Omega\cup\Omega_c} k^2\mathbf{w} \cdot \mathbf{E} dV = 0. \tag{6}$$

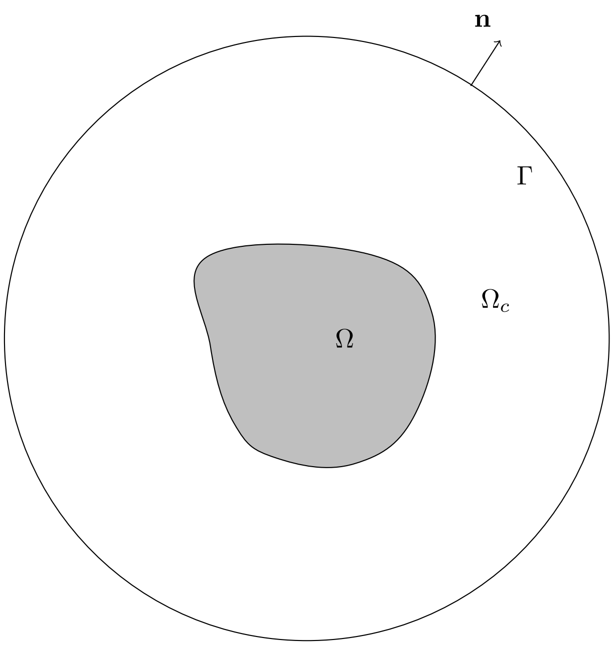

Figure 1: Domains of the problem.

Surface integrals over $\partial\Omega$ would formally arise. However, we require $\nabla \times \mathbf{E}$ to be tangentially continuous (H-field tangentially continuous). Hence such integrals don't appear. By substituting Eq. (2) to the boundary integral, we obtain

$$\int_{\Omega\cup\Omega_c} \nabla \times \mathbf{w} \cdot \nabla \times \mathbf{E} dV - ik \int_\Gamma \mathbf{w} \cdot \mathbf{E} dS - \int_{\Omega\cup\Omega_c} k^2 \mathbf{w} \cdot \mathbf{E} dV = \int_\Gamma \mathbf{w} \cdot \mathbf{g} dS. \tag{7}$$

We seek solutions in the form

$$\mathbf{E} = \sum_n \alpha_n \mathbf{x}_n \tag{8}$$

where $\mathbf{x}_n$ are curl-conforming basis functions, that are tangential to Γ, of a finite-dimensional subspace of $\mathbf{X}$. Then

$$\sum_n \alpha_n \Big(\int_{\Omega\cup\Omega_c} \nabla \times \mathbf{w}_m \cdot \nabla \times \mathbf{x}_n dV - ik \int_\Gamma \mathbf{w}_m \cdot \mathbf{x}_n dS - \int_{\Omega\cup\Omega_c} k^2 \mathbf{w}_m \cdot \mathbf{x}_n dV \Big) = \int_\Gamma \mathbf{w}_m \cdot \mathbf{g} dS. \tag{9}$$

for test functions $\mathbf{w}_m$.

The extinction cross-section is

$$\sigma_e = -\frac{1}{2} \Re \int_\Gamma (\mathbf{E}_0 \times \mathbf{H}_s^* + \mathbf{E}_s \times \mathbf{H}_0^*) \cdot \mathbf{n} dS \tag{10}$$

$$= \frac{1}{2} \Re \int_\Gamma \left(\frac{1}{i\omega\mu} \mathbf{E}_0 \times \nabla \times \mathbf{E}^* + \mathbf{E} \times \mathbf{H}_0^* \right) \cdot \mathbf{n} dS. \tag{11}$$